Convex Quadratic Programming in Graphs: links between continuous and discrete problems

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Outline

Stable sets and cliques

Throughout this presentation we deal with **simple graphs** (that is, graphs without loops and without parallel edges). These **simple graphs** are designated just by **graphs**.

Stable sets and cliques

A graph G with vertex set $V(G) = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and edge set $E(G) = \{12, 23, 36, 13, 34, 45, 56, 57, 58\}$.

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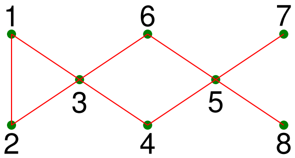

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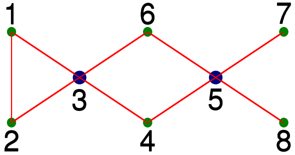

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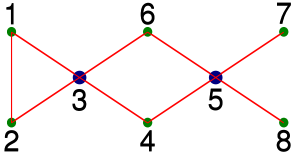

A stable set $S = \{3, 5\}$ and a clique $K = \{1, 2, 3\}$.

A maximum stable set

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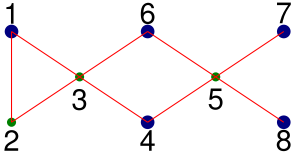

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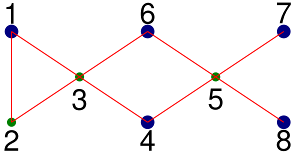
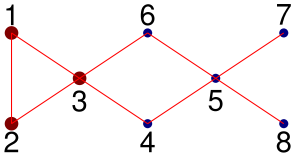


A maximum stable set S of G

$$S = \{1, 4, 6, 7, 8\} \Rightarrow \alpha(G) = 5$$

A maximum clique



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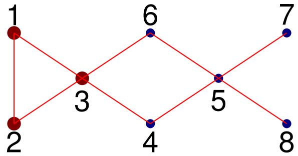

A maximum clique K of G

$$K = \{1, 2, 3\} \Rightarrow \omega(G) = 3$$

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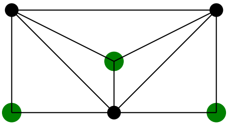

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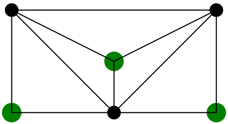

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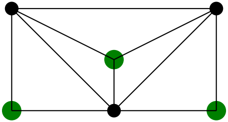

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The stability number, $\alpha(G)$, is the cardinality of maximum stable set of G.

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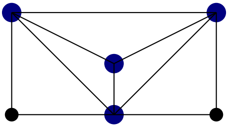

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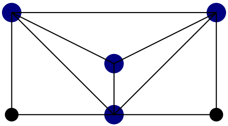

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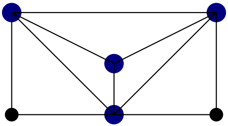

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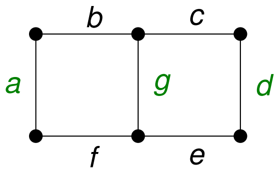

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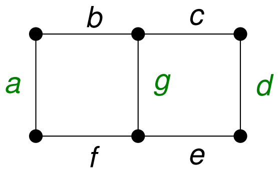

- A **perfect matching** is a matching M such that each vertex $v \in V(G)$ is incident in one edge of M.

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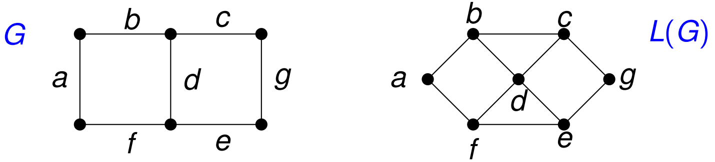

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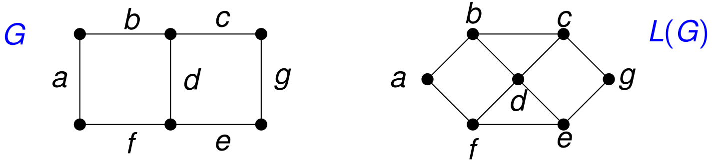

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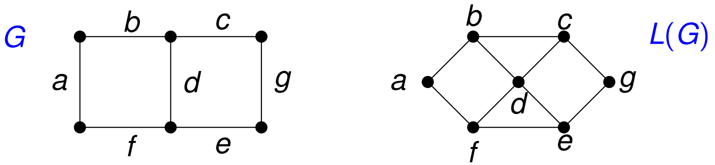

- The graph G has the perfect matching $\{a, d, g\}$ and then $L(G)$ has the maximum stable set $\{a, d, g\}$.

The complement of a graph

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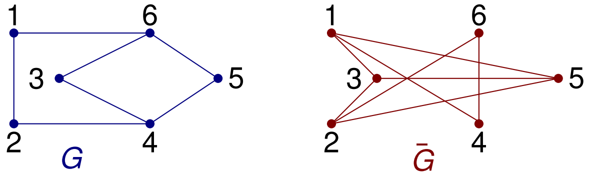

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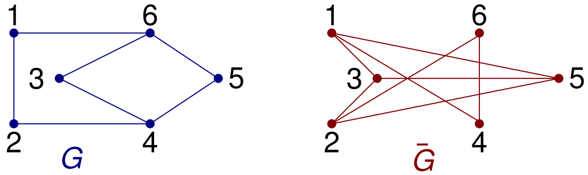

- Then $\alpha(G) = \omega(\bar{G})$ and determine the stability number is equivalent to determine the clique number.

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Then the **maximum stable set problem** is NP-hard in the class of H-free graphs.

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- There are classes of graphs for which the **maximum stable set problem** can be solved in polynomial time:
  - Claw-free graphs, which includes the line-graphs [(Berge, 1957), (Minty, 1980), (Sbihi, 1980)].
  - Particular subclasses of P_5-free graphs [(Mosca, 1997), (Mosca, 1999)], including $(P_5, K_{1,m})$-free graphs, $(P_5, K_{2,3})$-free graphs and (P_6, C_4)-free graphs.
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- Thus A_G is symmetric and it has n real eigenvalues

$$\lambda_{max}(A_G) = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = \lambda_{min}(A_G).$$

Basic spectral properties of a graph

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$$\lambda_{min}(A_G) \leq -1.$$

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- Denoting the minimum and maximum degree of the vertices of a graph G by $\delta(G)$ and $\Delta(G)$, respectively,

$$\delta(G) \leq \bar{d}_G \leq \lambda_{max}(A_G) \leq \Delta(G),$$

where $\bar{d}_G$ is the average degree of the vertices of G.

The Motzkin-Straus quadratic program

- Consider a graph G and the quadratic program

$$f(G) = \max\{\frac{1}{2}x^T A_G x : x \in \Delta\},$$

where $\Delta = \{x \geq 0 : \hat{e}^T x = 1\}$ and $\hat{e}^T = (1, 1, \dots, 1)$.

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Theorem[Motzkin-Straus, 1965]

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- From (1), after some algebraic manipulation,

$$\frac{1}{\alpha(G)} = \min\{x^T(A_G + I)x : x \in \Delta\}. \tag{2}$$

A family of quadratic programs

- Consider the families of quadratic programs (with $\tau > 0$):

$$\nu_G(\tau) = \min_{x \in \Delta} x^T\left(\frac{A_G}{\tau} + I\right)x, \tag{3}$$

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Theorem[C, 2003]

If x^* and y^* are optimal solutions for (3) and (4), respectively, then $\frac{x^*}{\nu_G(\tau)}$ and $\frac{y^*}{\upsilon_G(\tau)}$ are optimal solutions of (4) and (3), respectively. Furthermore, $\upsilon_G(\tau) = \frac{1}{\nu_G(\tau)}$.

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- As a consequence of this theorem, $\upsilon_G(1) = \alpha(G)$.

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  - $\forall U \subset V(G)\ \upsilon_{G-U}(\tau) \leq \upsilon_G(\tau)$.

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Theorem[Luz, 1995]

Let G be a graph with at least one edge. Then $\upsilon(G) = \alpha(G)$ iff for a maximum stable set $S \subset V(G)$ (and then for all)

$$-\lambda_{min}(A_G) \leq |N_G(v) \cap S| \quad \forall v \in V(G) \setminus S.$$

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■ If $E(G) \neq \emptyset$ and $\tau \geq -\lambda_{min}(A_G)$, then the quadratic program $\upsilon_G(\tau)$ is convex. The optimal value $\upsilon_G(-\lambda_{min}(A_G))$ was firstly introduced as an upper bound on $\alpha(G)$ in (Luz, 1995). $\upsilon_G(-\lambda_{min}(A_G))$ is simple denoted $\upsilon(G)$.

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Let G be a graph with at least one edge. Then $\upsilon(G) = \alpha(G)$ iff for a maximum stable set $S \subset V(G)$ (and then for all)

$$-\lambda_{min}(A_G) \leq |N_G(v) \cap S| \quad \forall v \in V(G) \setminus S.$$

■ $\upsilon(G) = \alpha(G)$ iff there exists a stable set $S \subset V(G)$ such that the above inequality holds (C. and Cvetković, 2006).

Optimality condition for $\upsilon(G)$

Theorem[C., 2001]

Let a_G^i be the i-th row of the matrix A_G. Then the n-tuple of real numbers x^* is an optimal solutions for the convex quadratic program $\upsilon(G)$ iff $\forall i \in V(G)$

$$x_i^* = \max\{0, 1 - r_i(x^*)\},$$

where $r_i(x^*) = \frac{a_G^i x^*}{\lceil -\lambda_{min}(A_G) \rceil}$.

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It is immediate that $\upsilon(G) = \alpha(G)$ iff $\upsilon(G)$ has a $0-1$ optimal solution x^* and in such a case x^* is the characteristic vector of a maximum stable set. Therefore, the above optimality conditions can be used, in a combinatorial way, to find (or to conclude that does not exists) a $0-1$ optimal solution.

Graphs with convex *QP*-stability number

- A graph G with at least one edge such that $\upsilon(G) = \alpha(G)$ is designated graph with **convex *QP*-stability number**, where *QP* means quadratic program.

Graphs with convex QP-stability number

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Graphs with convex *QP*-stability number

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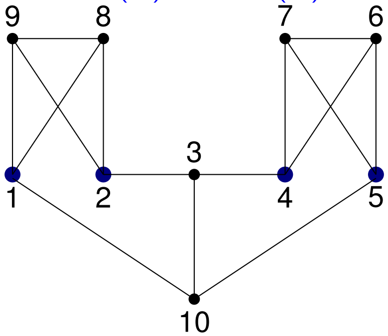
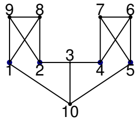
The cubic graph G, depicted in the next figure, is a Q-graph with $\lambda_{min}(A_G) = -2$ and $\upsilon(G) = 4 = \alpha(G)$.



An application of the discrete optimality condition

- Let us apply the discrete optimality conditions to the following graph.



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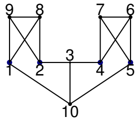
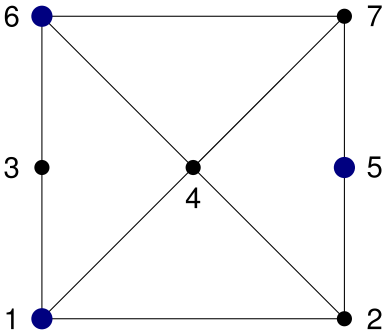


	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	$1 - r(x)$
x_1	1							0	0	0	1
x_2	1	1	0					0	0	0	1
x_3	1	1	0	1				0	0	0	0
x_4	1	1	0	1		0	0	0	0	0	1
x_5	1	1	0	1	1	0	0	0	0	0	1
x_6	1	1	0	1	1	0	0	0	0	0	0
x_7	1	1	0	1	1	0	0	0	0	0	0
x_8	1	1	0	1	1	0	0	0	0	0	0
x_9	1	1	0	1	1	0	0	0	0	0	0
x_{10}	1	1	0	1	1	0	0	0	0	0	0

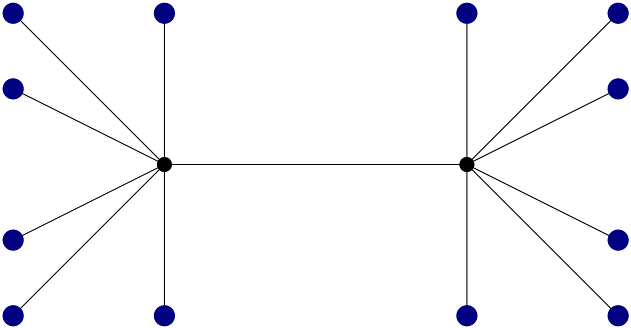
Additional examples of Q-graphs

A Q-graph with $\lambda_{min}(A_G) = -2$ and $\upsilon_G(2) = 3 = \alpha(G)$.



Additional examples of Q-graphs

A Q-graph with $\lambda_{min}(A_G) = -3$ and $\upsilon_G(3) = 12 = \alpha(G)$.



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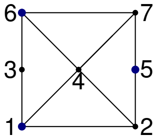

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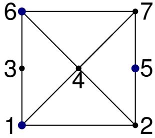

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$1 - r(x)$
x_1	1	0	0	0				1
x_2	1	0	0	0	1			1
x_3	1	0	0	0	1	1		1
x_4	1	0	0	0	1	1	1	$-1/2$
x_5	1	0	0	0	1	1	1	$1/2$
x_4	1	0	0	0	1	1	0	0
x_5	1	0	0	0	1	1	0	0
x_6	1	0	0	0	1	1	0	0
x_7	1	0	0	0	1	1	0	0

Some properties of Q-graphs

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then $G - U$ is a Q-graph.
- The following properties appear in (C,2001).
  - A G is a Q-graph iff each component is a Q-graph.
  - There exists an infinite number of Q-graphs (C, 2001).
  - A connected graph with at least one edge, which is nor a star neither a triangle, has a perfect matching if and only if its line graph is a Q-graph.
  - If each component of G has a nonzero even number of edges then $L(L(G))$ is a Q-graph.

Recognition of Q-graphs

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then G is not a Q-graph.

- Consider that $\exists v \in V(G)$ such that

$$\upsilon(G - \{v\}) \neq \upsilon(G - N_G(v)).$$

1. If $\upsilon(G) = \upsilon(G - \{v\})$ then G is a Q-graph iff $G - \{v\}$ is a Q-graph.
2. If $\upsilon(G) = \upsilon(G - N_G(v))$ then G is a Q-graph iff $G - N_G(v)$ is a Q-graph.

Recognition of Q-graphs (cont.)

- We have problems when $\forall v \in V(G)$
  - $\upsilon(G) = \upsilon(G - v) = \upsilon(G - N_G(v))$ and
  - $\lambda_{min}(A_G) = \lambda_{min}(A_{G-v}) = \lambda_{min}(A_{G-N_G(v)})$.

Recognition of Q-graphs (cont.)

- The graphs having an induced subgraph G without isolated vertices such that $\upsilon(G)$ is integer and $\forall v \in V(G)$

  1. $\upsilon(G) = \upsilon(G - N_G(v))$,
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Recognition of Q-graphs (cont.)

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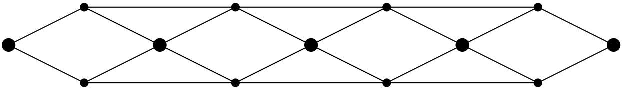
  are called **adverse graphs.**

- The graph G depicted in the next figure is an adverse graph (which is a Q-graph, since $\upsilon(G) = 5 = \alpha(G)$).



The adverse graph conjecture

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Conjecture

Every adverse graph is a Q-graph.

(κ, τ)-regular sets

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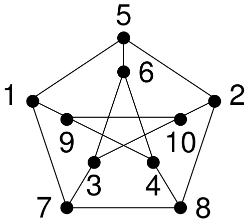

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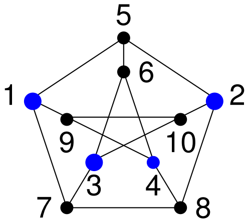

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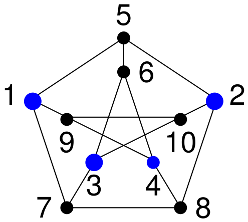

$S_1 = \{1, 2, 3, 4\}$ is $(0, 2)$-regular.

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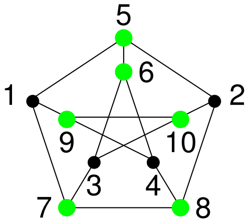

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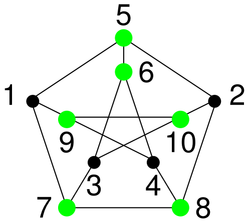

$S_2 = \{5, 6, 7, 8, 9, 10\}$ is $(1, 3)$-regular.

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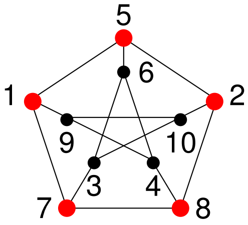

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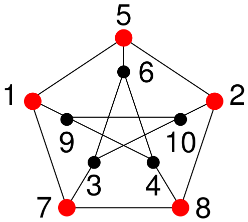

$S_3 = \{1, 2, 5, 7, 8\}$ is $(2, 1)$-regular.

A few properties

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- A p-regular graph G of order n is strongly regular with parameters (n, p, a, c) iff for every vertex $v \in V(G)$, the vertex subset
$$S = N_G(v)$$
is (a, c)-regular in $G - v$ [C, Sciriha, Zerafa, 2010].

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- Equivalent to the above statement, we may say that a p-regular graph G of order n is strongly regular with parameters (n, p, a, c) iff for every vertex $v \in V(G)$, the vertex subset
$$S = V(G) \setminus \{v\} \cup N_G(v))$$
is $(p - c, p - a - 1)$-regular in $G - v$.

(κ, τ)-regular sets and adverse graphs

Theorem[C., 2003]

An adverse graph G is a Q-graph iff $\exists S \subseteq V(G)$ which is $(0, \tau)$-regular, with $\tau = -\lambda_{min}(A_G)$.

(κ, τ)-regular sets and adverse graphs

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- Let G be a graph with at least one edge and consider the modified convex quadratic program on a parameter k,

$$\upsilon_k(G) = \max_{x \geq 0} 2\hat{e}^T x - \frac{\tau}{k+\tau} x^T (\frac{A_G}{\tau} + I_n) x,$$

where $\tau = -\lambda_{min}(A_G)$.

(κ, τ)-regular sets and k-regular induced subgraphs

Theorem[C., Kamiński and Lozin, 2007]

Let G be a graph of order n. If $\exists S \subset V(G)$ induces a subgraph H such that $\bar{d}_H = k$ (where $\bar{d}_H$ denotes the average degree of H, then $|S| \leq \upsilon_k(G)$.

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Theorem[C., Kamiński and Lozin, 2007]

If $\exists S \subseteq V(G)$ inducing a k-regular subgraph, then $|S| = \upsilon_k(G)$ iff $k + \tau \leq |N_G(v) \cap S| \; \forall v \in V(G) \setminus S$.

Bibliography

- V. E. Alekseev, On influence of local restrictions to the complexity of finding the independence number of the graph, Combinatorial and Algebraic Methods in Applied Mathematics, Gorkie University Press, Gorky, (1982): 3-13 (in Russian).
- C. Berge, *Two theorems in graph theory*. Proc. Nat. Acad. Sci. USA 43 (1957): 842-844.
- R. Barbosa, D. M. Cardoso, On regular-stable graphs, Ars Combinatoria, 70 (2004): 149-159.
- D. M. Cardoso, Convex Quadratic Programming Approach to the Maximum Matching Problem, Journal of Global Optimizathttp 21 (2001):91-106.

Bibliography

- D. M. Cardoso, On graphs with stability number equal to the optimal value of a convex quadratic program, *Matemática Contemporânea*, 25 (2003): 9-24.
- D. M. Cardoso, D. Cvetkovic, Graphs with least eigenvalue -2 attaining a convex quadratic upper bound for the stability number, Bull. Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur. Sci. Math., 31 (2006): 41-55.
- D. M. Cardoso, M. Kaminski, V. Lozin, Maximum k-regular induced subgraphs. Journal of Combinatorial Optimization, Vol. 14, n. 4 (2007): 455-463.
- D. M. Cardoso, I. Sciriha, C. Zerafa, "Main eigenvalues and (k, τ)-regular sets." Linear Algebra Appl. 423 (2010): 2399-2408.

Bibliography

D. M. Cardoso, V. V. Lozin, On hereditary properties of the class of graphs with convex quadratic stability, Journal of Mathematical Sciences, Vol. 182, No. 2, (2012): 227-232.

R. M. Karp, Reducibility among combinatorial problems, In: *Complexity of Computer Computations*, eds. R. E. Miller and J. W. Thatcher, Plenum Press, New York,(1972): 85-104.

C. J. Luz, An upper bound on the independence number of a graph computable in polynomial time, *Operations Research Letters*, 18 (1995): 139-145.

G. J. Minty, On maximal independent sets of vertices in claw-free graphs, *J. Comb. Theory* Ser. B 28 (1980): 284-304.

Bibliography

R. Mosca, Polynomial algorithms for the maximum independent set problem on particular classes of P_5-free graphs, *Inform. Process. Lett.* 61 (1997): 137-144.

R. Mosca, Independent sets in certain P_6-free graphs, Discrete Appl. Math. 92 (1999): 177-191.

T. S. Motzkin and E. G. Straus, Maxima for graphs and a new proof of a theorem of Túran, Canadian Journal of Mathematics, 17 (1965): 533-540.

N. Sbihi, Algorithm de recherche d'un independent de cardinalité maximum dans une graphe sans étoiles, *Discrete Mathematics* 29 (1980): 53-76.